Electric Circuit Element Boundary Conditions in the Finite Element Method for Full-Wave Frequency Domain Passive Devices

Gabriela Ciuprina, Daniel Ioan, Mihai Popescu, and Sorin Lup

Abstract A natural coupling of a circuit with an electromagnetic (EM) device is possible if special boundary conditions, called Electric Circuit Element (ECE), are used for the EM field formulation. This contribution shows how these ECE boundary conditions can be implemented into the finite element method for the solving of coupled full-wave EM field-circuit problems in the frequency domain. The implementation is based on a weak formulation that uses the electric field strength strictly inside the domain and a scalar potential defined solely on the boundary. Edge elements are used inside the three-dimensional domain and nodal elements are used on its two-dimensional boundary surface. The weak formulation is given and its discrete form is validated on a 2D example, with known analytic solution.

1 Motivation

Many EM devices with distributed parameters and field effects specific to full-wave (FW) or Magneto-Quasi-Static (MQS) EM field regime are connected to circuits with lumped parameters (e.g. in measuring and control applications). For this, the EM devices need boundary conditions compatible with external circuits (Fig. 1,left).

By definition, an isolated electric circuit has a finite number of components connected to common terminals. Each terminal is characterized by its voltage with respect to the ground. A non-isolated circuit, i.e. a sub-circuit with m terminal nodes has each of these terminals characterized by a pair of scalar quantities, a current i_k entering into the sub-circuit and a "node voltage" (potential) v_k (Fig. 1-right). The power transferred to it is

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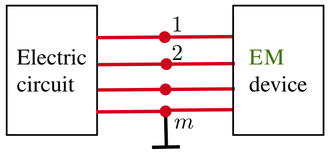

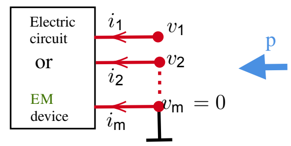

Fig. 1 Left: Coupling of electric circuits and EM device models are naturally ensured by means of terminals. Right: To ensure the coupling, "node voltages" (potentials) and electric currents of non-isolated circuits must have a correspondent in the EM device model.

$$P = \sum_{k=1}^{m} i_k v_k = \sum_{k=1}^{m-1} i_k (v_k - v_m) = \sum_{k=1}^{m-1} i_k v_k \tag{1}$$

if i_m is expressed according to Kirchhoff current law for a cutset and the terminal m is connected to ground. This power expression shows that the state of a m-terminal circuit is characterized by $2(m-1)$ independent quantities: $m-1$ currents and $m-1$ voltages. The assumption $v_m = 0$ is not a restriction for the purpose of this paper, which is stated at the end of Section 2. A natural coupling of this sub-circuit with an EM device is possible if some connecting surfaces are defined on the device's boundary, for which currents and potentials are defined, in order to satisfy Kirchhoff relationships and provide the same transmitted power formula (1) as subcircuits do. The conditions that satisfy these requirements are the ones proposed in [10], used in [4,8] and called Electric Circuit Element (ECE) boundary conditions.

The ECE boundary conditions, combined with current excited terminals, are the "realistic boundary conditions" used in [2] to solve eddy current problems with the finite element method (FEM) using a formulation in $\mathbf{H}$ and an ungauged $\mathbf{T} - \varphi, \varphi$ one in [1]. Similar conditions, although with a different definition for the terminal voltages are proposed in [6] and used for $\mathbf{A}, V$ eddy current formulations [5].

The use of ECE in MQS problems for inductance extraction with an $\mathbf{A}, V$ formulation is discussed in [9]. Our aim is to use ECE boundary conditions to solve full-wave (FW) problems with FEM. We have successfully used ECE to model passive on-chip components such as resistors, inductors, capacitors, interconnects or RF-MEMS switches in FW [3], with the Finite Integration Technique as numerical method. According to our knowledge, the ECE conditions are not available in FEM codes which implement the formulation of microwave ports for FW. Theoretical studies exists, e.g. in [4], based on an $\mathbf{E}, V$ formulation for the whole domain. In this paper we use $\mathbf{E}$ strictly inside the domain and V solely on the boundary. During the reviewing process of this paper, Hiptmair and Ostrowski released a relevant report [7], proving the interest for this subject.

2 ECE Boundary conditions

Assume a simply connected domain Ω, with a Lipschitz boundary $\partial\Omega$ that includes m disjoint parts S_k, $k = 1, 2, \ldots, m$ (device's terminals), so that conditions (ECE1),

(ECE2) and (ECE3) are satisfied:
• **(ECE1)** there is no magnetic coupling with the exterior: $\mathbf{n} \cdot \frac{\partial \mathbf{B}(\mathbf{r},t)}{\partial t} = 0, \quad \forall \mathbf{r} \in \partial \Omega$;
• **(ECE2)** the electric coupling is carried out only through the terminals:
$\mathbf{n} \cdot (\nabla \times \mathbf{H}(\mathbf{r},t)) = 0, \quad \forall \mathbf{r} \in \partial \Omega - \cup_{k=1}^{m} S_k$;
• **(ECE3)** the terminals are equipotential: $\mathbf{n} \times \mathbf{E}(\mathbf{r},t) = \mathbf{0}, \quad \forall \mathbf{r} \in S_k,\ k = 1, \ldots, m$.
According to Faraday's law, (ECE3) implies (ECE1) for the terminals, the inclusion of the terminals in (ECE1) is kept only for emphasizing the physical meaning.

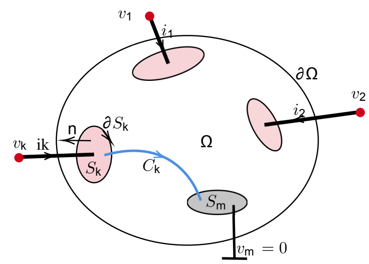

Fig. 2 Electric terminals are disjoint surfaces on the domain's boundary. The non-grounded terminals can be either voltage excited (its potential is given) or current excited (its total current is given).

By definition, the currents and potentials of any terminal are:

$$i_k(t) = \oint_{\partial S_k} \mathbf{H} \cdot \mathbf{dl} = -\int_{S_k} \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \right) \cdot \mathbf{n}\, \mathrm{d}s, \qquad v_k(t) = \int_{C_k \subset \partial \Omega} \mathbf{E} \cdot \mathbf{dl}, \tag{2}$$

where, in order to ensure conservation, each terminal current is the total current (conductive and displacement) and the potential is properly defined as the voltage between this terminal and the reference one, along a path C_k included in the domain boundary. Due to (ECE1) the voltage between two points placed on the boundary surface is independent of the path of the integration line connecting these points, provided that this path is included in the surface. Thus, the potential on the surface is well defined, although this is not the case in a general time-varying EM field. Under these conditions, (1) holds for the EM device, where i_k and v_k are given by (2), and thus the ECE boundary conditions are perfectly compatible with the power transferred through its terminals by a multipolar circuit [8, 10].

If we assume that the terminals have known potentials, then it can be proved that the problem of EM field analysis in a linear domain with ECE boundary conditions has a unique solution. Consequently, the terminal currents are output signals and are obtained by solving the field problem [10]. As the domain is linear, so are the equations, hence the device with ECE conditions is a linear system, defining a multiple input multiple output (MIMO) type dynamic system with $m-1$ inputs and $m-1$ outputs (Fig. 3).

In the frequency domain, the input-output relationship is expressed as:

$$\left[\underline{V}_1 \ \ldots \ \underline{V}_n \ \underline{I}_{n+1} \ \ldots \ \underline{I}_{m-1} \right]^T = \begin{bmatrix} \underline{\mathbf{Z}} & \underline{\mathbf{A}} \\ \underline{\mathbf{B}} & \underline{\mathbf{Y}} \end{bmatrix} \left[\underline{I}_1 \ \ldots \ \underline{I}_n \ \underline{V}_{n+1} \ \ldots \ \underline{V}_{m-1} \right]^T . \tag{3}$$

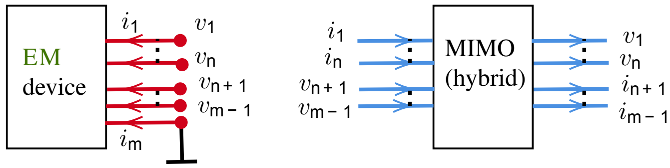

Fig. 3 Each non-grounded terminal of the EM device with ECE boundary conditions can be either current excited or voltage excited. Its hybrid transfer matrix is obtained after computing voltages of the current excited terminals and currents of the voltage excited terminals in linear problems.

The problem to be solved is: "Find $\begin{bmatrix} \underline{\mathbf{Z}}(f) & \underline{\mathbf{A}}(f) \\ \underline{\mathbf{B}}(f) & \underline{\mathbf{Y}}(f) \end{bmatrix}$, where f is the frequency in a given frequency range of interest, defined by its minimum and maximum values $f_{\min}$ and $f_{\max}$ $f \in [f_{\min}, f_{\max}]$, from the EM field solution." If this hybrid matrix is known, then the "field" element can be realized with common circuit elements and included in any circuit simulator.

3 ECE in FEM

It is useful to recall the formulation in $\mathbf{E}$ with classical boundary conditions, since the newly proposed formulation inherits a part of it.

Strong formulation of PDE for E with classical boundary conditions. The well known FW Maxwell equations in the frequency domain, for linear media and no internal field sources are: $\nabla \times \underline{\mathbf{E}} = -\mathrm{j}\omega\mu\underline{\mathbf{H}}$, $\nabla \times \underline{\mathbf{H}} = \sigma\mathbf{E} + \mathrm{j}\omega\varepsilon\underline{\mathbf{E}}$, $\nabla \cdot (\mu\underline{\mathbf{H}}) = 0$, $\nabla \cdot (\varepsilon\underline{\mathbf{E}}) = \rho$, where permittivity ε, permeability μ and conductivity σ are positive, space dependent material parameters. The reluctivity $\nu = 1/\mu$ might be used instead of μ. The solution of these equations is unique if in any point of $\partial\Omega$, either exclusively $\underline{\mathbf{E}}_t$ or $\underline{\mathbf{H}}_t$ are known (given). The subscript t indicates the tangential component of the vector on the surface. It is useful to denote a disjoint partition of the boundary: $\partial\Omega = S_E \cup S_H$, $S_E \cap S_H = \emptyset$, and thus $\underline{\mathbf{E}}_t : S_E \to \mathbb{C}^2$, $\underline{\mathbf{H}}_t : S_H \to \mathbb{C}^2$. The imposed boundary conditions are: $\underline{\mathbf{E}}_t(\mathbf{r}) = \mathbf{n} \times (\mathbf{E}(\mathbf{r}) \times \mathbf{n})$, for $\mathbf{r} \in S_E$ and $\underline{\mathbf{H}}_t(\mathbf{r}) = \mathbf{n} \times (\mathbf{H}(\mathbf{r}) \times \mathbf{n})$, for $\mathbf{r} \in S_H$. In what follows we will name them ***classical boundary conditions***. The uniqueness of the field solution can be proven on the basis of the complex form of the Poynting's theorem that gives the expression of the transmitted power (assuming a linear field domain, with no moving parts):

$$-\oint_{\partial\Omega} (\underline{\mathbf{E}}_t \times \underline{\mathbf{H}}_t^*) \cdot \mathbf{n}\, \mathrm{d}s = \int_{\Omega} \underline{\mathbf{E}} \cdot \underline{\mathbf{J}}^* + 2\mathrm{j}\omega \int_{\Omega} \left(\frac{\underline{\mathbf{B}} \cdot \underline{\mathbf{H}}^*}{2} - \frac{\underline{\mathbf{E}} \cdot \underline{\mathbf{D}}^*}{2} \right). \tag{4}$$

The proof assumes that there exist two such fields that satisfy the same boundary conditions. This means that the Poynting theorem in complex form is valid for the difference field, which satisfies Maxwell's equations (due to linearity) and zero boundary conditions. This implies that the real part is zero which conduces to zero difference electric field (conductivity of the domain is assumed non-zero everywhere) and the imaginary part is zero with conduces to zero difference magnetic field.

The second order equation is:

$$\nabla \times (\nu \nabla \times \underline{\mathbf{E}}) + \mathrm{j}\omega(\sigma + \mathrm{j}\omega\varepsilon)\underline{\mathbf{E}} = \mathbf{0}. \tag{5}$$

Weak formulation in E with classical boundary conditions.
In general, solving of (5) implies a numerical approach, e.g. FEM, which is based on weak formulations. The needed functionals result by projecting (5) onto a set of test functions $\underline{\mathbf{E}}'$, then integrating by parts and applying Gauss-Ostrogradski formula:

$$\int_{\Omega} \left[(\nu \nabla \times \underline{\mathbf{E}}) \cdot (\nabla \times \underline{\mathbf{E}}') + \mathrm{j}\omega(\sigma + \mathrm{j}\omega\varepsilon)\underline{\mathbf{E}} \cdot \underline{\mathbf{E}}' \right] \mathrm{d}x = -\oint_{\partial\Omega} \left[(\nu \nabla \times \underline{\mathbf{E}}) \times \underline{\mathbf{E}}' \right] \cdot \mathbf{n}\, \mathrm{d}s$$

Replacing the expression of the magnetic field strength in the right hand side we get

$$\int_{\Omega} \left[(\nu \nabla \times \underline{\mathbf{E}}) \cdot (\nabla \times \underline{\mathbf{E}}') + \mathrm{j}\omega(\sigma + \mathrm{j}\omega\varepsilon)\underline{\mathbf{E}} \cdot \underline{\mathbf{E}}' \right] \mathrm{d}x = \mathrm{j}\omega \oint_{\partial\Omega} \left(\underline{\mathbf{H}} \times \underline{\mathbf{E}}' \right) \cdot \mathbf{n}\, \mathrm{d}s. \tag{6}$$

With classical boundary conditions, the right hand side is equal to $\int_{S_E} (\underline{\mathbf{E}}'_t \times \mathbf{n}) \cdot \underline{\mathbf{H}}\, \mathrm{d}s + \int_{S_H} (\mathbf{n} \times \underline{\mathbf{H}}_t) \cdot \underline{\mathbf{E}}'\, \mathrm{d}s$. $\underline{\mathbf{E}}_t$ are essential boundary conditions that is why the test functions are chosen so that $\underline{\mathbf{E}}'_t$ is zero on S_E. Thus, the weak equation for the trial functions $\underline{\mathbf{E}}$ is:

$$\int_{\Omega} \left[(\nu \nabla \times \underline{\mathbf{E}}) \cdot (\nabla \times \underline{\mathbf{E}}') + \mathrm{j}\omega(\sigma + \mathrm{j}\omega\varepsilon)\underline{\mathbf{E}} \cdot \underline{\mathbf{E}}' \right] \mathrm{d}x = \mathrm{j}\omega \int_{S_H} (\mathbf{n} \times \underline{\mathbf{H}}_t) \cdot \underline{\mathbf{E}}'\, \mathrm{d}s. \tag{7}$$

The boundary conditions $\underline{\mathbf{H}}_t$ are natural, they appear in the functional equation.

In conclusion, the weak formulation in $\underline{\mathbf{E}}$ with classical boundary conditions is:
Find $\underline{\mathbf{E}}$ in $\mathscr{H}$, such that $a(\underline{\mathbf{E}}, \underline{\mathbf{E}}') = f(\underline{\mathbf{E}}')$, $\quad \forall \underline{\mathbf{E}}' \in \mathscr{H}_0$ where

$$a((\underline{\mathbf{E}}, \underline{\mathbf{E}}') = \int_{\Omega} \left[(\nu \nabla \times \underline{\mathbf{E}}) \cdot (\nabla \times \underline{\mathbf{E}}') + \mathrm{j}\omega(\sigma + \mathrm{j}\omega\varepsilon)\underline{\mathbf{E}} \cdot \underline{\mathbf{E}}' \right] \mathrm{d}x, \tag{8}$$

$$f(\underline{\mathbf{E}}') = \mathrm{j}\omega \int_{S_H} (\mathbf{n} \times \underline{\mathbf{H}}_t) \cdot \underline{\mathbf{E}}'\, \mathrm{d}s, \tag{9}$$

$$\mathscr{H} = \{ \mathbf{u} \in \mathscr{H}(\mathrm{curl}, \Omega) | \mathbf{n} \times (\mathbf{u} \times \mathbf{n}) = \underline{\mathbf{E}}_t \quad \text{on} \quad S_E \}, \tag{10}$$

$$\mathscr{H}_0 = \{ \mathbf{u} \in \mathscr{H}(\mathrm{curl}, \Omega) | \mathbf{n} \times (\mathbf{u} \times \mathbf{n}) = \mathbf{0} \quad S_E \}. \tag{11}$$

Discrete formulation in E with classical boundary conditions.
Assume a simplicial mesh (tetrahedrons in 3D, triangles in 2D), numerical test functions $\mathbf{N}_k$ that correspond to edge elements of order (0,1), and degrees of freedom that represent the complex representations of voltages $\underline{U}_k$ along the edges. The numerical solution is approximated as $\underline{\mathbf{E}} = \sum_{j=1}^{\mathrm{Ne}} \underline{U}_j \mathbf{N}_j$, where Ne is the total number of edges in the domain. For any cell, the sum involves 6 terms in 3D and 3 terms in 2D. By substituting the approximation of the numerical solution in (6), choosing the test function $\underline{\mathbf{E}}' = \mathbf{N}_i$ and rearranging the sums we obtain a relationship that reveals how the matrices assembling has to be done for all $i = 1, \ldots, \mathrm{Ne}$:

$$\sum_{j=1}^{\mathrm{Ne}} \left\{ \int_{\Omega} \left[(\nu \nabla \times \mathbf{N}_j) \cdot (\nabla \times \mathbf{N}_i) + \mathrm{j}\omega(\sigma + \mathrm{j}\omega\varepsilon) \mathbf{N}_j \cdot \mathbf{N}_i \right] \mathrm{d}x \right\} \underline{U}_j = \mathrm{j}\omega \int_{S_H} (\mathbf{n} \times \underline{\mathbf{H}}_t) \cdot \mathbf{N}_i \, \mathrm{d}s. \tag{12}$$

The initial assembling is carried out for all the edges in the domain. The next step refers to the boundary conditions. Assume that the edges were numbered in the following order: first - the inner edges, second - the edges on the boundary S_H and finally, the edges on the boundary S_E. This leads to the following partitioning:

$$\begin{bmatrix} \mathbf{A}_{\mathrm{in-in}} & \mathbf{A}_{\mathrm{in-SH}} & \mathbf{A}_{\mathrm{in-SE}} \\ \mathbf{A}_{\mathrm{SH-in}} & \mathbf{A}_{\mathrm{SH-SH}} & \mathbf{A}_{\mathrm{SH-SE}} \\ \mathbf{A}_{\mathrm{SE-in}} & \mathbf{A}_{\mathrm{SE-SH}} & \mathbf{A}_{\mathrm{SE-SE}} \end{bmatrix} \begin{bmatrix} \underline{\mathbf{U}}_{\mathrm{in}} \\ \underline{\mathbf{U}}_{\mathrm{SH}} \\ \underline{\mathbf{U}}_{\mathrm{SE}} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{b}_{\mathrm{SH}} \\ \mathbf{0} \end{bmatrix} \tag{13}$$

The group of equations that correspond to edges on the S_E boundary is deleted and the essential boundary conditions $\underline{\mathbf{E}}_t$ are translated into imposed values of electric voltages along edges on the S_E boundary. The system to be solved is

$$\begin{bmatrix} \mathbf{A}_{\mathrm{in-in}} & \mathbf{A}_{\mathrm{in-SH}} \\ \mathbf{A}_{\mathrm{SH-in}} & \mathbf{A}_{\mathrm{SH-SH}} \end{bmatrix} \begin{bmatrix} \underline{\mathbf{U}}_{\mathrm{in}} \\ \underline{\mathbf{U}}_{\mathrm{SH}} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{b}_{\mathrm{SH}} \end{bmatrix} - \begin{bmatrix} \mathbf{A}_{\mathrm{in-SE}} \\ \mathbf{A}_{\mathrm{SH-SE}} \end{bmatrix} \left[\underline{\mathbf{U}}_{\mathrm{SE}} \right], \tag{14}$$

the coefficient matrix being symmetric and positive defined.

Weak formulation in $\mathbf{E}, V$ with ECE boundary conditions.

If we use ECE boundary conditions, the unknowns are the electric field inside the domain and an electric scalar potential solely defined on $\partial\Omega$. That is why the formulation is still named $\mathbf{E}, V$, but is different from other formulations, such as the $\mathbf{E}, V$ in [4] where V is defined also inside the domain. An $\mathbf{E}, V$ interpretation of the ECE boundary conditions (ECE 1,2,3)) is:

- **(ECE1b)** $\oint_{\Gamma} \underline{\mathbf{E}} \cdot \mathbf{dl} = 0, \quad \forall \Gamma \in \partial\Omega;$
- **(ECE2b)** $\mathbf{n} \cdot \underline{\mathbf{E}}(\mathbf{r}) = 0, \quad \forall \mathbf{r} \in \partial\Omega - \cup_{k=1}^{m} S_k;$
- **(ECE3b)** $\underline{\mathbf{E}}_t(\mathbf{r}) = \mathbf{0} \quad \forall \mathbf{r} \in S_k,\ k = 1, \ldots, m.$

From (ECE1b) an electric scalar potential $\underline{V}$ can be defined on the boundary $\partial\Omega$, such that $\underline{\mathbf{E}}_t = -\nabla_2 \underline{V}$. Condition (ECE3b) requires that the electric terminals are equipotential. For uniqueness reasons, one terminal has to be defined by any value. Without lack of generality we can assume it is grounded in what follows. For the other terminals the uniqueness implies that, exclusively, either their voltages or currents are known.

Using (5) we get the weak equation for $\mathbf{E}$:

$$\int_{\Omega} \left[(\nu \nabla \times \underline{\mathbf{E}}) \cdot (\nabla \times \underline{\mathbf{E}}') + \mathrm{j}\omega(\sigma + \mathrm{j}\omega\varepsilon) \underline{\mathbf{E}} \cdot \underline{\mathbf{E}}' \right] \mathrm{d}x = \mathrm{j}\omega \sum_{k \in \mathscr{I}_c} \underline{V}'_k \underline{I}_k, \tag{15}$$

where $\mathscr{I}_c$ is the set of indices of current excited terminals. Similarly, we will denote by $\mathscr{I}_v$ is the set of indices of voltage excited terminals. We need an equation for the electric potential on the boundary, as well. Let's denote the normal component of the total current density in any point on the boundary as $\underline{J}_n \stackrel{\mathrm{not}}{=} (\nabla \times \underline{\mathbf{H}}) \cdot \mathbf{n}$. We will project $\underline{J}_n$ onto a set of scalar test functions $\underline{V}'$:

$$\oint_{\partial\Omega} (\nabla \times \underline{\mathbf{H}}) \cdot \mathbf{n}\underline{V}' \,\mathrm{d}s = \oint_{\partial\Omega} \underline{J}_n \underline{V}' \,\mathrm{d}s \overset{\text{(ECE2)}}{=} \sum_{k=1}^{m} \int_{S_k} \underline{J}_n \underline{V}' \,\mathrm{d}s = \sum_{k \in \mathscr{I}_c} \underline{V}'_k \underline{I}_k$$

The integrand of the left hand side can be further computed by using the integration by parts formula that involves the surface differential operators and the substitution of the magnetic field with its expression with respect to the electric field, as it follows from Faraday's law:

$$\oint_{\partial\Omega} (\nabla \times \underline{\mathbf{H}}) \cdot \mathbf{n}\underline{V}' \,\mathrm{d}s = \oint_{\partial\Omega} \underline{V}' \mathbf{n} \cdot \mathrm{curl}\, \underline{\mathbf{H}} \,\mathrm{d}s \overset{\text{def}}{=} \oint_{\partial\Omega} \underline{V}' \,\mathrm{div}_2\, (\underline{\mathbf{H}}) \,\mathrm{d}s =$$
$$= \int_{\partial(\partial\Omega)} \underline{V}' (\mathbf{n} \times \mathbf{H}) \,\mathrm{d}s - \oint_{\partial\Omega} \underline{\mathbf{H}} \cdot \mathrm{grad}_2 \underline{V}' \,\mathrm{d}s = \oint_{\partial\Omega} \frac{\nu}{j\omega} \mathrm{curl}\, \underline{\mathbf{E}} \cdot \mathrm{grad}_2 \underline{V}' \,\mathrm{d}s$$

Consequently it follows that the weak form of the equation on the boundary is

$$\oint_{\partial\Omega} (\nu \nabla \times \underline{\mathbf{E}}) \cdot \nabla_2 \underline{V}' \,\mathrm{d}s = j\omega \sum_{k \in \mathscr{I}_c} \underline{V}'_k \underline{I}_k \tag{16}$$

Finally, we get the weak formulation in $\underline{\mathbf{E}}, V$ with ECE boundary conditions.

Find $\underline{\mathbf{E}} \in \mathscr{H}_E$, $\underline{V} \in \mathscr{H}_V$, such that

$$a(\underline{\mathbf{E}}, \underline{\mathbf{E}}') = f(\underline{\mathbf{E}}'), \quad \forall \underline{\mathbf{E}}' \in \mathscr{H}_{E,0}; \qquad b(\underline{\mathbf{E}}, \underline{V}') = g(\underline{V}'), \quad \forall \underline{V}' \in \mathscr{H}_{V,0}$$
$$\oint_{\partial S_k} \mathbf{H} \cdot \mathbf{dl} = \underline{I}_k, \quad k \in \mathscr{I}_c; \qquad \underline{\mathbf{E}}_t = -\nabla_2 V, \quad \text{on} \quad \partial\Omega,$$

where

$$a(\underline{\mathbf{E}}, \underline{\mathbf{E}}') = \int_{\Omega} \left[(\nu \nabla \times \underline{\mathbf{E}}) \cdot (\nabla \times \underline{\mathbf{E}}') + \mathrm{j}\omega(\sigma + \mathrm{j}\omega\varepsilon) \underline{\mathbf{E}} \cdot \underline{\mathbf{E}}' \right] \mathrm{d}x, \qquad f(\underline{\mathbf{E}}') = \mathrm{j}\omega \sum_{k \in \mathscr{I}_c} \underline{V}'_k \underline{I}_k;$$
$$b(\underline{\mathbf{E}}, \underline{V}') = \oint_{\partial\Omega} (\nu \nabla \times \underline{\mathbf{E}}) \cdot \nabla_2 \underline{V}' \,\mathrm{d}s, \qquad g(\underline{V}') = \mathrm{j}\omega \sum_{k \in \mathscr{I}_c} \underline{V}'_k \underline{I}_k;$$

where $\underline{\mathbf{E}}'_t = -\nabla_2 V'$.

$$\begin{aligned}
\mathscr{H}_E = \{\mathbf{u} \in \mathscr{H}(\mathrm{curl}, \Omega) \,|\, & \mathbf{n} \times (\mathbf{u} \times \mathbf{n}) = -\nabla_2 \underline{V}' \text{ on } \partial\Omega, \quad \underline{V}' \in \mathscr{H}_V \\
& \mathbf{n} \times (\mathbf{u} \times \mathbf{n}) = \mathbf{0} \text{ on } \cup_{k=1}^{m} S_k\} \\
\mathscr{H}_{E,0} = \{\mathbf{u} \in \mathscr{H}(\mathrm{curl}, \Omega) \,|\, & \mathbf{n} \times (\mathbf{u} \times \mathbf{n}) = -\nabla_2 \underline{V}' \text{ on } \partial\Omega, \quad \underline{V}' \in \mathscr{H}_{V,0} \\
& \mathbf{n} \times (\mathbf{u} \times \mathbf{n}) = \mathbf{0} \text{ on } \cup_{k=1}^{m} S_k\} \\
\mathscr{H}_V = \{u \in \mathscr{H}(\mathrm{grad}, \partial\Omega) \,|\, & u = \underline{V}_k \text{ on } S_k,\ k \in \mathscr{I}_v, \\
& u = \mathrm{constant(unkown)} \text{ on } S_k,\ k \in \mathscr{I}_c\ \}
\end{aligned}$$

$$\begin{aligned}
\mathscr{H}_{V,0} = \{u \in \mathscr{H}(\mathrm{grad}, \partial\Omega) \,|\, & u = 0 \text{ on } S_k,\ k \in \mathscr{I}_v \\
& u = \mathrm{constant(unkown)} \text{ on } S_k,\ k \in \mathscr{I}_c\ \}
\end{aligned}$$

Note: We have investigated two other formulations for the boundary equations for which $b(\mathbf{E},V')=0$. In one version $b(\mathbf{E},V')=\oint_{\partial\Omega}(\sigma+\mathrm{j}\omega\varepsilon)(\nabla_2\underline{V})\cdot(\nabla_2\underline{V}')\,\mathrm{d}s+\oint_{\partial\Omega}\frac{\partial}{\partial n}[(\sigma+\mathrm{j}\omega\varepsilon)\mathbf{E}\cdot\mathbf{n}]\underline{V}'\,\mathrm{d}s$ and another version is $b(\mathbf{E},V')=\oint_{\partial\Omega}(\sigma+j\omega\varepsilon)\mathbf{n}\cdot\mathbf{E}V'\,\mathrm{d}s$. Due to lack of space we will not present them here.

Formulation in $\mathbf{E},V$ with ECE boundary conditions - algorithm in FEM.

Step 1: We start with the discrete form of classical BC, given by (14), written for all the edges) $\begin{bmatrix}\mathbf{A}_{u,u} & \mathbf{A}_{u,u_b}\\ \mathbf{A}_{u_b,u} & \mathbf{A}_{u_b,u_b}\end{bmatrix}\begin{bmatrix}\mathbf{u}\\ \mathbf{u}_b\end{bmatrix}=\begin{bmatrix}\mathbf{0}\\ \mathbf{b}_b\end{bmatrix}$. Only the first block row of equations, corresponding to the inner edges, is kept.

Step 2: Write the discrete form of the equation (16) on the 2D surface boundary mesh. $\sum_{j=1}^{Ne}\left[\oint_{\partial\Omega}(\nu\nabla\times\mathbf{N}_j)\cdot(\nabla_2\underline{\varphi_i}')\,\mathrm{d}s\right]\underline{U_j}=j\omega\underline{I}_i$, where $\underline{\varphi_i}'$ is the nodal element i. This is written for all the nodes on the boundary and will be placed together with the discrete equation obtained at step 1: $\begin{bmatrix}\mathbf{A}_{u,u} & \mathbf{A}_{u,u_b}\\ \mathbf{A}_{V_b,u} & \mathbf{A}_{V_b,u_b}\end{bmatrix}\begin{bmatrix}\mathbf{u}\\ \mathbf{u}_b\end{bmatrix}=\begin{bmatrix}\mathbf{0}\\ \mathbf{b}'_b\end{bmatrix}$.

Step 3: On the boundary, the variables are changed, from electric voltages to electric potentials, by expressing $\mathbf{u}_b$ as potential differences. The system becomes $\begin{bmatrix}\mathbf{A}_{u,u} & \mathbf{A}_{u,V_b}\\ \mathbf{A}_{V_b,u} & \mathbf{A}_{V_b,V_b}\end{bmatrix}\begin{bmatrix}\mathbf{u}\\ \mathbf{V}_b\end{bmatrix}=\begin{bmatrix}\mathbf{0}\\ \mathbf{b}'_b\end{bmatrix}$.

Step 4: Finally, $\mathbf{V}_b$ has to be split in three ($\mathbf{V}$-for nodes that are not on terminals, $\mathbf{V}_{t,c}$-voltages of current excited terminals, $\mathbf{V}_{t,v}$- voltages of voltage excited terminals), in order to impose the rest of the natural conditions (potentials for voltage excited, or currents for current excited terminals): Finally, the system to solve is

$$\begin{bmatrix}\mathbf{A}_{u,u} & \mathbf{A}_{u,V} & \mathbf{A}_{u,V_{t,c}}\\ \mathbf{A}_{V,u} & \mathbf{A}_{V,V} & \mathbf{A}_{V,V_{t,c}}\\ \mathbf{A}_{V_{t,c},u} & \mathbf{A}_{V_{t,c},V} & \mathbf{A}_{V_{t,c},V_{t,c}}\end{bmatrix}\begin{bmatrix}\mathbf{u}\\ \mathbf{V}\\ \mathbf{V}_{t,c}\end{bmatrix}=\begin{bmatrix}\mathbf{0}\\ \mathbf{0}\\ j\omega\mathbf{I}_{t,c}\end{bmatrix}-\begin{bmatrix}\mathbf{A}_{u,V_{t,v}}\\ \mathbf{A}_{V,V_{t,v}}\\ \mathbf{A}_{V_{t,c},V_{t,v}}\end{bmatrix}\begin{bmatrix}\mathbf{V}_{t,v}\end{bmatrix}.$$

After solving, we get the unknown potentials $\mathbf{V}$ and $\mathbf{V_{t,c}}$. The currents through the terminals in $\mathscr{I}_v$ can be computed as a postprocessing step.

3.1 Numerical results

Fig. 4 shows a quantitative validation for a 2D simple case, with two terminals and with analytical solution. It is a single input single output (SISO) system, both current and voltage excitations give accurate results. The domain is a brick that occupies the space $x\in[-a,a]$, $y\in[0,l]$ and $x\in[0,h]$. One excited terminal (in voltage or in current) is on the $z=0$ boundary and the grounded terminal is on the $z=h$ boundary. The material inside is assumed homogeneous with ε,μ,σ. The analytic solution can be obtained by solving the Helmholtz equations and considering the current excited terminal ($\underline{I}$). The complex power absorbed by this domain is $\underline{P}=2\underline{E}_y\underline{H}_z^*\,lh$, where $\underline{E}_y=\underline{\gamma}/(\sigma+j\omega\varepsilon)\cosh(\underline{\gamma}a)/\sinh(\underline{\gamma}a)\,\underline{I}/(2h)$ and $\underline{H}_z=\underline{I}/(2h)$. The extracted complex impedance is $\underline{Z} = \underline{P}/|\underline{I}|^2$ and its components shown in Fig. 4 are $R = \mathrm{real}\,\underline{P}$ and $L = \mathrm{real}\,\underline{P}/\omega$ for $a = 2.5\,\mu$m, $l = 10\,\mu$m, $h = 10\,\mu$m, $\sigma = 6.6 \cdot 10^7$ S/m, $\mu = \mu_0$, $\varepsilon = \varepsilon_0$, $f_{min} = 0.01$ GHz, $f_{max} = 100$ GHz.

Fig. 5 shows a qualitative validation for a MIMO test. The rectangular domain is occupied by a T-shape conductor of high conductivity, having 3 terminals, out of which the one at the right hand side of the figure is grounded.

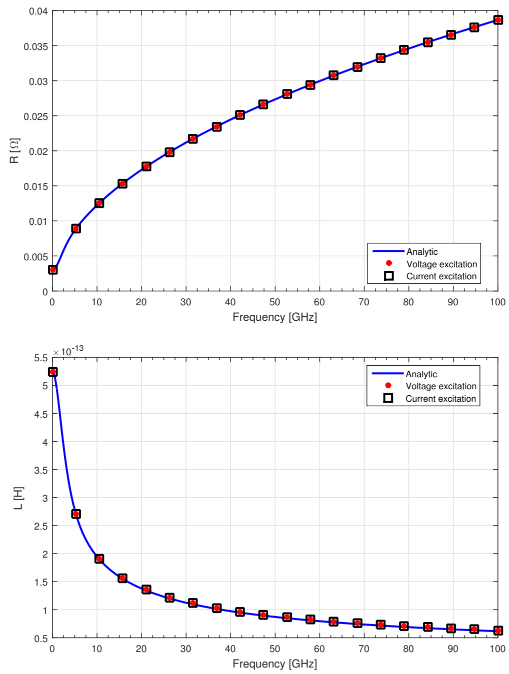

Fig. 4 Quantitative validation of the implementation for a 2D case with analytical solution. The problem is a rectangle with two opposite terminals, consequently the system is SISO. Both voltage and current excitations lead to relative errors less than 2% for the whole frequency range.

4 Conclusions

The advantages of ECE BC for Maxwell equations are that the ports are clearly and well defined, without ambiguity, fully compatible with the circuit terminals. There is no restriction on the field regime (full wave, nonlinear). For MIMO systems, the hybrid excitation is obtained in a natural way. This paper proposed a FEM algorithm for ECE, which **E** strictly inside the domain and V on the boundary. The degrees of freedom are the electric voltages on the inner edges and the potentials of the floating nodes on the boundary (nodes outside terminals and current excited terminals). Our next research will compare the 3 mentioned formulations.

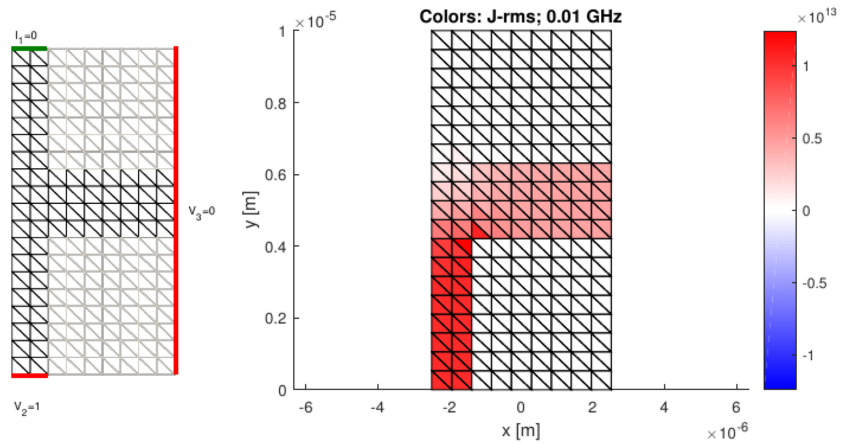

Fig. 5 Qualitative validation for a 2D case, MIMO (3 terminals), hybrid excitation (one terminal grounded, one is voltage excited and one is current excited).

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